Solution to
Finding Fundamental Matrix for Stereo Vision

Afshin Sepehri

We find the fundamental matrix of stereo vision using singular value decomposition, rank enforcement and normalization for the provided corresponding points.

> **Problem 1.** Find the coefficient vector a as a function of values x_l, y_l, x_r and y_r.

Answer:

$$p_r^T F p_l = 0$$

$$\begin{bmatrix} x_r & y_r & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_r F_{11} + y_r F_{21} + F_{31} & x_r F_{12} + y_r F_{22} + F_{32} & x_r F_{13} + y_r F_{23} + F_{33} \end{bmatrix} \begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_r x_l & x_r y_l & x_r & y_r x_l & y_r y_l & y_r & x_l & y_l & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

$$a = \begin{bmatrix} x_r x_l & x_r y_l & x_r & y_r x_l & y_r y_l & y_r & x_l & y_l & 1 \end{bmatrix}$$

> **Problem 2.** Find the matrix A using the points provided in two matrices `Pl` and `Pr` in file `points.mat`. Coordinates of corresponding points can be found in the same rows of these two matrices, one point in a row. We refer to these two matrices as P_l and P_r.

Answer: See Matlab file `FindMatrixA.m`

> **Problem 3.** Suppose we have SVD of matrix A, that is
>
> $$A = UDV^T \tag{1}$$
>
> Prove that the best approximation to f, is the column of V corresponding to the least singular value of A.

Answer:

$$\underset{(m\times n)}{A} = \underset{(m\times m)}{U}\ \underset{(m\times n)}{D}\ \underset{(n\times n)}{V^T} \qquad (m \geq n)$$

$$A = \left[\begin{array}{cccc|ccc} u_1 & u_2 & \dots & u_n & u_{n+1} & \dots & u_m \end{array}\right] \left[\begin{array}{ccc} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline 0 & & \\ & \ddots & \\ & & 0 \end{array}\right] \left[\begin{array}{c} v_1^T \\ \vdots \\ v_n^T \end{array}\right]$$

$$A = \sigma_1 u_1 v_1^T + \cdots + \sigma_n u_n v_n^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T$$

The best solution to $Af = 0$ can be found by

$$\min_{\|f\|=1} \|Af\|^2$$

where

$$\|Af\|^2 = \|\sum_{i=1}^{n} \sigma_i u_i (v_i^T f)\|^2 = \sum_{i=1}^{n} (\sigma_i (v_i^T f))^2 \|u_i\|^2$$

Since $\|u_i\|^2 = 1$, then

$$\|Af\|^2 = \sum_{i=1}^{n} (\sigma_i (v_i^T f))^2$$

If we consider $f = v_k$ where σ_k is the least singular value of A, since we have

$$v_k^T v_j = 0 \qquad (k \neq j)$$

and

$$v_k^T v_k = 1$$

Therefore:

$$\min_{\|f\|=1} \|Af\|^2 = \sigma_k^2$$

which is the minimum value of $\|Af\|$ as all the components of the summation are nonnegative. As a result, $f = v_k$ is the best solution to $Af = 0$.

Problem 4. Find SVD of matrix A as (1) and find the best approximation to f based upon the theorem of problem 3. Do not forget to scale the result vector f such that the last element gets value 1.

Build fundamental matrix F from the elements of vector f and find the norm of the error vector containing $p_r^T F p_l$ for all corresponding points p_l and p_r in matrices P_l and P_r.

Answer: See Matlab files `FindBestF.m` and `FindErrorNorm.m`

Problem 5. Find SVD of matrix F, that is

$$F = UDV^T$$

Set the smallest singular value in the diagonal of D equal to 0, let D' be the corrected matrix.

Find the corrected estimate of F, F', as

$$F' = UD'V^T$$

Find the norm of the error vector containing $p_r^T F' p_l$ for all corresponding points p_l and p_r in matrices P_l and P_r and compare the result with result of problem 4. Verify that matrix F' has rank 2.

Answer: See Matlab file `FindRank2F.m`

Comparison: As expected, the error we get from $p_r^T F' p_l$ is higher than that of $p_r^T F p_l$, because the elements of matrix F differ from the solution we found from minimization process. However, we also expect that for some random points on the image, the new matrix F gives us lower error, because the true fundamental matrix should be of rank 2.

Problem 6. Find matrices H_l and H_r using the points given in P_l and P_r and as described above. Verify that the average norm of each component of $\widehat{p_i}$ is almost 1.

Answer: See Matlab file `NormalizePoints.m`

Problem 7. Prove that fundamental matrix F can be found as

$$F = H_r^T \widehat{F} H_l \tag{2}$$

Answer:

$$\widehat{p_r}^T \widehat{F} \widehat{p_l} = 0$$

$$(H_r p_r)^T \widehat{F} (H_l p_l) = 0$$

$$p_r^T(H_r^T\widehat{F}H_l)p_l = 0$$

By considering $F = H_r^T\widehat{F}H_l$,

$$p_r^T F p_l = 0$$

Problem 8. Repeat the steps of problems 2, 4 and 5 using the normalized points $\widehat{P_l}$ and $\widehat{P_r}$. Find new fundamental matrix F using (2) and compare it with the one found in problem 5.

Answer: Use a combination of above Matlab files. Note that rank enforcement should be done prior to multiplying matrices H_l and H_r to $\widehat{F}$.

Comparison: By finding the norm of the error vector, we see that the result matrix F is slightly better than that of problem 5. Meanwhile, this amendment makes the algorithm more stable.

Problem 9. Find L_r, the matrix of epipolar lines where each row is the corresponding epipolar line for a point in P_l. Test if all these lines intersect in a single point and explain how you tested it.

Answer: See Matlab file `FindEpipolarLines.m`

Test Method: To test whether all the computed epipolar lines intersect in a single point or not, we can use the similar method as we used to find best f in problem 3 (*i.e.* SVD). As a result of the test, the least singular value of matrix L_r is very close to zero showing that all the lines intersect almost in a single point. The intersection point is called *epipole*. Note that in the provided images, since the image planes of the two cameras are almost parallel, the epipoles locate outside the boundary of the images.

Problem 10. Prove that fundamental matrix F_{rl} can be found as

$$F_{rl} = F_{lr}^T$$

Answer:

$$(p_r^T F_{lr} p_l)^T = 0$$
$$p_l^T F_{lr}^T p_r = 0$$

So, by changing the role of p_l and p_r, we can find

$$F_{rl} = F_{lr}^T$$

Final Word: You can see the aggregated solution for all problems in the file `Solution.m`. Also, you can compute the fundamental matrix using a set of corresponding points and by calling function `FindFundamentalMatrix.m`.